# Knot and Prime

We first look at circle $S^1$

finite cyclic cover: $\mathbb{R}/n\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$ $\left( \mathrm{Gal}(\mathbb{R}/S^1) = \{ \rho \in \mathrm{Aut}(\mathbb{R}) \mid \begin{smallmatrix} \mathbb{R} \xrightarrow{\rho} \mathbb{R} \\ \searrow \ \swarrow \\ S^1 \end{smallmatrix} \} \right.$

generator of $\underset{\mathrm{Gal}(\mathbb{R}/S^1)}{\overset{\pi_1(S^1)}{\parallel}}$, $\iota: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x+1$.

Tubular neighborhood $V := S^1 \times D^2$ ( $D^2$ is 2-dim disk)

$V$ is homotopy equivalent to $S^1$

$V \setminus S^1$ is homotopy equivalent to $\partial V$

$S^1$ is homotopically the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$

Now look at finite field $\mathbb{F}_p$ ( $p$ is a prime).

finite cyclic cover. $\mathbb{F}_{p^n}/\mathbb{F}_p$ ( deg $n$ field extension)

generator of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ : Frobenius automorphism $Fr: x \longmapsto x^p$

( Here $\bar{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_q$ )

If I say $\mathbb{F}_p$ is the analogy of $S^1$. you may not feel excited. No strong evidence suggests that it is right analogy and nothing interested happens here.

## § Some algebra geometry

We have a contravariant functor.

$\text{Spec}$ : Category of ring $\longrightarrow$ Category of topology

$$A \longmapsto \text{Spec}(A)$$

$\text{Spec}\, A = \{$ points are prime ideals in $A$ with "some topology" $\}$

Eg

$$\text{Spec}\,\mathbb{Z} = \{(0), (2), (3), (5), (7), \cdots\}$$

Now we can compare.

circle $\quad S^1 \quad \longleftrightarrow \quad \text{Spec}\,\mathbb{F}_p$

Cover $\quad \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \quad \longleftrightarrow \quad$ étale cover: $\text{Spec}\,\mathbb{F}_{p^n} \to \text{Spec}\,\mathbb{F}_p$

$\pi_1(S^1) \cong \text{Gal}(\mathbb{R}/S^1) = \mathbb{Z} \quad \longleftrightarrow \quad \pi_1(\mathbb{F}_q) \cong \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}$

$\hat{\mathbb{Z}}$ || profinite completion of $\mathbb{Z}$ $\varprojlim \mathbb{Z}/n\mathbb{Z}$

generator $\quad l : \mathbb{R} \to \mathbb{R},\ a \to a+1 \quad \longleftrightarrow \quad$ generator: $Fr : a \to a^p$

Usual topology $\longleftrightarrow$ étale topology

Now, you may still not believe me that $\mathbb{F}_q$ is an anology of $S^1$ since all I do here is to write everything in a fancy way. But you can predict what can happen in a number theory world by looking at knots theory. We keep looking at other relative object.

Recall

Tubular $\quad V = S^1 \times D^2 \quad \longleftrightarrow \quad \mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ (p-adic integer)

$V$ is homotopy equivalent to $S^1 \quad \longleftrightarrow \quad$ Fact: $\text{Spec}\,\mathbb{Z}_p$ is homotopy equivalent to $\text{Spec}\,\mathbb{F}_p$

$V \setminus S^1$ is homotopy equivalent to $\partial V$ $\longleftrightarrow$ Fact: $\mathrm{Spec}(\mathbb{Z}_p) \setminus \mathrm{Spec}\,\mathbb{F}_p = \mathrm{Spec}\,\mathbb{Q}_p$

(Here $\mathbb{Q}_p$ is fractional field of $\mathbb{Z}_p$)

Though the right side may be abstract nonsense to you. Now you can predict what should happen in number theory world by looking at knot theory side. Let's say we consider the following map.

$$\pi_1(\partial V) \longrightarrow \pi_1(V) = \pi_1(S^1) \longleftrightarrow \pi_1(\mathrm{Spec}\,\mathbb{Q}_p) \longrightarrow \pi_1(\mathrm{Spec}\,\mathbb{Z}_p) \simeq \pi_1(\mathrm{Spec}\,\mathbb{F}_p).$$

The preimage of $l \in \pi_1(S^1)$ is $\beta = S^1 \times \{b\}$ $(b \in \partial D^2)$ $\longleftrightarrow$ preimage of $F_r \in \pi_1(\mathrm{Spec}\,\mathbb{F}_p)$ is Frobenius automorphism, denoted $\sigma$

kernel is infinite cyclic gp generated by $\alpha = \{a\} \times \partial D^2$ $(a \in S^1)$ $\longleftrightarrow$ kernel $I_p^{\text{tame}}$ is called inertial group. generated by a monodromy $\tau$.

$\pi_1(\partial V)$ is generated by $\alpha$ and $\beta$. $\longleftrightarrow$ $\pi_1^{\text{tame}}(\mathrm{Spec}\,\mathbb{Q}_p)$ is generated by $\sigma$ and $\tau$.

$$\pi_1(\partial V) = \langle \alpha, \beta \mid [\alpha, \beta] = 1 \rangle \longleftrightarrow \pi_1^{\text{tame}}(\mathrm{Spec}\,\mathbb{Q}_p) = \langle \sigma, \tau \mid \tau^{p-1}[\tau, \sigma] = 1 \rangle$$

Here we haven't talked anything about knots. we only look at circle $S^1$

knot: $S^1 \hookrightarrow \mathbb{R}^3$ (or 3-dim manifold $M$) $\longleftrightarrow$ $\mathrm{Spec}\,\mathbb{F}_p \hookrightarrow \mathrm{Spec}\,\mathbb{Z}$ (or $\mathrm{Spec}\,\mathcal{O}_K$)

$M$ has cohomological dimension 3 $\longleftrightarrow$ $\mathrm{Spec}\,\mathcal{O}_K$ has cohomological dimension 3.

Poincaré duality for singular cohomology $\longleftrightarrow$ Poincaré duality in étale cohomology.

Let $K$ be a knot in a closed $M$ ($S^3$)

knot group: $G_K = \pi_1(M \setminus K)$ $\longleftrightarrow$ Galois group: $G_{\{p\}} = \pi_1(\operatorname{Spec}(\mathcal{O}_k) \setminus \{p\})$

$G_{\{p\}} = \pi_1(\operatorname{Spec} \mathbb{Z} \setminus \{p\})$.

$G_K = \pi_1(S^3 \setminus K)$

= Galois group of maximal extension of $k$ unramified outside prime $p$

Prop (Whitten, Gordon-Luecke)

for knots $K$ and $L$ in $S^3$ $\longleftrightarrow$ For prime $p$ and $q$.

$G_K \cong G_L \iff K \cong L$ (up to orientation) $\qquad$ $G_{\{p\}} \cong G_{\{q\}} \iff p = q$.

peripheral group $D_K = \pi_1(\partial V_K)$ $\longleftrightarrow$ Decomposition group $\pi_1(\operatorname{Spec}(\mathcal{O}_k) \setminus \{p\})$

$= \pi_1(\partial(M \setminus K))$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$G_K \qquad\qquad\qquad\qquad G_{\{p\}}$$

Homology group $\qquad$ idea class group.

linking number $\qquad$ Legendre symbols

Milnor invariants $\qquad$ multiple power residue symbols

Infinite cyclic cover.

We have a knot $K \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$

Let $X = S^3 \setminus K$.

Let $\tilde{X}$ be an infinite cyclic cover of $X$.

i.e. $\operatorname{Aut}\left(\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & X & \end{tikzcd}\right) = G \cong \mathbb{Z}$

It is determined by some homomorphism from $\pi_1(X) \longrightarrow G \cong \mathbb{Z}$

Take a generator of $G$ such that $\begin{aligned} G &\cong \mathbb{Z} \\ t &\longmapsto 1 \end{aligned}$

Let $F$ be any field.
Then $t$ acts on $C_*(\tilde{X}, F)$, therefore $t$ acts on $H_i(\tilde{X}, F)$
Hence we can view $C_*(\tilde{X}, F)$ and $H_i(\tilde{X}, F)$ as group algebra $F[G]$ module

Fact: $C_*(\tilde{X}, F)$ is free and finitely generated over $F[G]$, with one generator for each $i$-cell of $X$.

Hence $H_i(\tilde{X}, F)$ is finitely generated $F[G]$ module.
Notice that $F[G] \cong F[t, t^{-1}]$ is principal domain.
We can study the structure of $H_i(\tilde{X}, F)$.
Fact. $H_i(\tilde{X}, F)$ is finitely generated torsion $F[G]$ module

In particularly
$$H_1(\tilde{X}, F) \cong F[G]/(P_1) \oplus F[G]/(P_2) \oplus \cdots \oplus F[G]/(P_k)$$

Def: Any generator of the idea $(P_1 P_2 \cdots P_k)$ is called Alexander polynomial

For number theory side, we have a tower of field extension.

$$F_0 \subset F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_\infty$$

$$\mathrm{Gal}(F_n/F_0) = \mathbb{Z}/p^n\mathbb{Z}. \quad F_\infty = \bigcup_n F_n$$

We can view $F_\infty$ as infinite cyclic cover of $F_0$.

Here $Gal(F_\infty / F_0) = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$. Hence we call $\mathbb{Z}_p$ extension of $F_0$

For each field, we can associated a class group.

p-part of $Cl(F_\infty)$ is also $\mathbb{Z}_p[[t]]$ finite generated module.

$$p\text{-part of } Cl(F_\infty) \sim \bigoplus_{i=1}^{k} \Lambda/(P_i)$$

Define characteristic polynomial $f$ as generator of idea $(P_1 P_2 \cdots P_k)$

$\lambda = \deg f$ is called Iwasawa lambda invariant.

My research is to build relation between $\lambda$ and Massey product. which comes from knots theory.

Reference.


Masanori Morishita. Knots and primes.

John W. Milnor. Infinite cyclic coverings.